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String theories as the adiabatic limit of Yang-Mills theory

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Abstract

We consider Yang-Mills theory with a matrix gauge group G on a direct product manifold $M = \Sigma_2 \times H^2$, where Σ_2 is a two-dimensional Lorentzian manifold and H^2 is a two-dimensional open disc with the boundary $S^1 = \partial H^2$. The Euler-Lagrange equations for the metric on Σ_2 yield constraint equations for the Yang-Mills energy-momentum tensor. We show that in the adiabatic limit, when the metric on H^2 is scaled down, the Yang-Mills equations plus constraints on the energy-momentum tensor become the equations describing strings with a worldsheet Σ_2 moving in the based loop group $\Omega G = C^\infty(S^1, G)/G$, where S^1 is the boundary of H^2. By choosing $G = \mathbb{R}^{d-1,1}$ and putting to zero all parameters in $\Omega\mathbb{R}^{d-1,1}$ besides $\mathbb{R}^{d-1,1}$, we get a string moving in $\mathbb{R}^{d-1,1}$. In arXiv:1506.02175 it was described how one can obtain the Green-Schwarz superstring action from Yang-Mills theory on $\Sigma_2 \times H^2$ while H^2 shrinks to a point. Here we also consider Yang-Mills theory on a three-dimensional manifold $\Sigma_2 \times S^1$ and show that in the limit when the radius of S^1 tends to zero, the Yang-Mills action functional supplemented by a Wess-Zumino-type term becomes the Green-Schwarz superstring action.

1. Introduction. Superstring theory has a long history [1]-[3] and pretends on description of all four known forces in Nature. In the low-energy limit superstring theories describe supergravity in ten dimensions or supergravity interacting with supersymmetric Yang-Mills (SYM) theory. On the other hand, Yang-Mills and SYM theories in four dimensions give description of three main forces in Nature not including gravity [4]-[7]. The aim of this short paper is to show that bosonic strings (both open and closed) as well as type I, IIA and IIB superstrings can be obtained as a subsector of pure Yang-Mills theory with some constraints on the Yang-Mills energy-momentum tensor. Put differently, knowing the action for superstrings with a worldsheet Σ_2, we introduce a Yang-Mills action functional on $\Sigma_2 \times H^2$ or on $\Sigma_2 \times S^1$ such that the Yang-Mills action becomes the Green-Schwarz superstring action while H^2 or S^1 shrink to a point. We will work in Lorentzian signature, but all calculations can be repeated for the Euclidean signature of spacetime.

2. Yang-Mills equations. Consider Yang-Mills theory with a matrix gauge group G on a direct product manifold $M = \Sigma_2 \times H^2$, where Σ_2 is a two-dimensional Lorentzian manifold (flat case is included) with local coordinates $x^a, a, b, ... = 1, 2$, and a metric tensor $g_{\Sigma_2} = (g_{ab})$, H^2 is the disc with coordinates x^i, $i, j, ... = 3, 4$, satisfying the inequality $(x^3)^2 + (x^4)^2 < 1$, and the metric $g_{H^2} = (g_{ij})$. Then $(x^\mu) = (x^a, x^i)$ are local coordinates on M with $\mu = 1, ..., 4$.

We start with the gauge potential $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$ with values in the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ having scalar product $(\cdot, \cdot)$ defined either via trace Tr or, for abelian groups like $\mathbb{R}^{p,q}$, $T^{p,q}$ etc., via a metric on vector spaces. The gauge field $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the $\mathfrak{g}$-valued two-form

$$\mathcal{F} = \tfrac{1}{2}\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \quad \text{with} \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \ . \tag{1}$$

The Yang-Mills equations on M with the metric

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{ab}\mathrm{d}x^a \mathrm{d}x^b + g_{ij}\mathrm{d}x^i \mathrm{d}x^j \tag{2}$$

have the form

$$D_\mu \mathcal{F}^{\mu\nu} := \frac{1}{\sqrt{|\det g|}}\, \partial_\mu (\sqrt{|\det g|}\, \mathcal{F}^{\mu\nu}) + [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = 0 \ , \tag{3}$$

where $g = (g_{\mu\nu})$ and $\partial_\mu = \partial / \partial x^\mu$.

The equations (3) follow from the standard Yang-Mills action on M

$$S = \tfrac{1}{4} \int_M \mathrm{d}^4 x \, \sqrt{|\det g|}\, (\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}) \ , \tag{4}$$

where $(\cdot, \cdot)$ is the scalar product on the Lie algebra $\mathfrak{g}$. Note that the metric g_{Σ_2} on Σ_2 is not fixed and the Euler-Lagrange equations for g_{Σ_2} yield the constraint equations

$$T_{ab} = g^{\lambda\sigma}(\mathcal{F}_{a\lambda}, \mathcal{F}_{b\sigma}) - \tfrac{1}{4}\, g_{ab}(\mathcal{F}_{\mu\nu}, \mathcal{F}^{\mu\nu}) = 0 \tag{5}$$

for the Yang-Mills energy-momentum tensor $T_{\mu\nu}$, i.e. its components along Σ_2 are vanishing. Note that these constraints can be satisfied for many gauge configurations, e.g. for self-dual gauge fields not only $T_{ab} = 0$ but even $T_{\mu\nu} = 0$.

3. Adiabatic limit. On $M = \Sigma_2 \times H^2$ we have the obvious splitting

$$\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu = \mathcal{A}_a \mathrm{d}x^a + \mathcal{A}_i \mathrm{d}x^i \ , \tag{6}$$

$$\mathcal{F} = \tfrac{1}{2}\mathcal{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \tfrac{1}{2}\mathcal{F}_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b + \mathcal{F}_{ai}\mathrm{d}x^a \wedge \mathrm{d}x^i + \tfrac{1}{2}\mathcal{F}_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j \ , \tag{7}$$

$$T = T_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = T_{ab}\mathrm{d}x^a\mathrm{d}x^b + 2T_{ai}\mathrm{d}x^a\mathrm{d}x^i + T_{ij}\mathrm{d}x^i\mathrm{d}x^j \ . \tag{8}$$

By using the adiabatic approach in the form presented in [8, 9], we deform the metric (2) and introduce the metric

$$\mathrm{d}s^2_\varepsilon = g_{ab}\mathrm{d}x^a\mathrm{d}x^b + \varepsilon^2 g_{ij}\mathrm{d}x^i\mathrm{d}x^j \ , \tag{9}$$

where $\varepsilon \in [0,1]$ is a real parameter. It is assumed that the fields $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ smoothly depend in ε^2, i.e. $\mathcal{A}_\mu = \mathcal{A}^{(0)}_\mu + \varepsilon^2\mathcal{A}^{(1)}_\mu + ...$ and $\mathcal{F}_{\mu\nu} = \mathcal{F}^{(0)}_{\mu\nu} + \varepsilon^2\mathcal{F}^{(1)}_{\mu\nu} + ...$. Furthermore, we have $\det g_\varepsilon = \varepsilon^4 \det(g_{ab})\det(g_{ij})$ and

$$\mathcal{F}^{ab}_\varepsilon = g^{ac}_\varepsilon g^{bd}_\varepsilon \mathcal{F}_{cd} = \mathcal{F}^{ab} \ , \quad \mathcal{F}^{ai}_\varepsilon = g^{ac}_\varepsilon g^{ij}_\varepsilon \mathcal{F}_{cj} = \varepsilon^{-2}\mathcal{F}^{ai} \quad \text{and} \quad \mathcal{F}^{ij}_\varepsilon = g^{ik}_\varepsilon g^{jl}_\varepsilon \mathcal{F}_{kl} = \varepsilon^{-4}\mathcal{F}^{ij} \ , \tag{10}$$

where indices in $\mathcal{F}^{\mu\nu}$ are raised by the non-deformed metric tensor $g^{\mu\nu}$.

For the deformed metric (9) the action functional (4) is changed to

$$S_\varepsilon = \tfrac{1}{4}\int_M \mathrm{d}^4x \sqrt{|\det g_{\Sigma_2}|}\sqrt{\det g_{H_2}} \left\{\varepsilon^2(\mathcal{F}_{ab},\mathcal{F}^{ab}) + 2(\mathcal{F}_{ai},\mathcal{F}^{ai}) + \varepsilon^{-2}(\mathcal{F}_{ij},\mathcal{F}^{ij})\right\} \ . \tag{11}$$

The term $\varepsilon^{-2}(\mathcal{F}_{ij},\mathcal{F}^{ij})$ in the Yang-Mills Lagrangian (11) diverges when $\varepsilon \to 0$. To avoid this we impose the flatness condition

$$\mathcal{F}^{(0)}_{ij} = 0 \quad \Rightarrow \quad \lim_{\varepsilon\to 0}(\varepsilon^{-1}\mathcal{F}_{ij}) = 0 \tag{12}$$

on the components of the field tensor along H^2. Here $\mathcal{F}^{(0)}_{ij} = 0$ but $\mathcal{F}^{(1)}_{ij}$ etc. in the ε^2-expansion must not be zero. For the deformed metric (9) the Yang-Mills equations have the form

$$\varepsilon^2 D_a\mathcal{F}^{ab} + D_i\mathcal{F}^{ib} = 0 \ , \tag{13}$$

$$\varepsilon\, D_a\mathcal{F}^{aj} + \varepsilon^{-1}D_i\mathcal{F}^{ij} = 0 \ . \tag{14}$$

In the deformed metric (9) the constraint equations (5) become

$$T^\varepsilon_{ab}{=}\varepsilon^2\left\{g^{cd}(\mathcal{F}_{ac},\mathcal{F}_{bd}){-}\tfrac{1}{4}g_{ab}(\mathcal{F}_{cd},\mathcal{F}^{cd})\right\}+g^{ij}(\mathcal{F}_{ai},\mathcal{F}_{bj}){-}\tfrac{1}{2}g_{ab}(\mathcal{F}_{ci},\mathcal{F}^{ci}){-}\tfrac{1}{4}\varepsilon^{-2}g_{ab}(\mathcal{F}_{ij},\mathcal{F}^{ij}){=}0. \tag{15}$$

In the adiabatic limit $\varepsilon \to 0$ the Yang-Mills equations (13) and (14) become

$$D_i\mathcal{F}^{ib} = 0 \ , \tag{16}$$

$$D_a\mathcal{F}^{aj} = 0 \ , \tag{17}$$

since the ε^{-1}-term vanishes due to (12). We also keep (17) since it follows from the action (11) after taking the limit $\varepsilon \to 0$. One can see that the constraint equations (15) are nonsingular in the limit $\varepsilon \to 0$ also due to (12):

$$T^0_{ab} = g^{ij}(\mathcal{F}_{ai},\mathcal{F}_{bj}) - \tfrac{1}{2}g_{ab}(\mathcal{F}_{ci},\mathcal{F}^{ci}) = 0 \ . \tag{18}$$

Note that for the adiabatic limit of instanton equations [8, 9] the constraints (15) disappear since the energy-momentum tensor for self-dual and anti-self-dual gauge fields vanishes on any four-manifold M.

4. Flat connections. Now we start to consider the flatness equation (12), the equations (16), (17) and the constraint equations (18). From now on we will consider only zero modes in ε^2-expansions and equations on them. For simplicity of notation we will omit the index "(0)" from all $\mathcal{A}^{(0)}$ and $\mathcal{F}^{(0)}$ tensor components. In the adiabatic approach it is assumed that all fields depend on coordinates $x^a \in \Sigma_2$ only via moduli parameters $\phi^\alpha(x^a), \alpha, \beta = 1, 2, ...$, appearing in the solutions of the flatness equation (12).

Flat connection $\mathcal{A}_{H^2} := \mathcal{A}_i \mathrm{d}x^i$ on H^2 has the form

$$\mathcal{A}_{H^2} = g^{-1}\hat{\mathrm{d}}g \quad \text{with} \quad \hat{\mathrm{d}} = \mathrm{d}x^i \partial_i \quad \text{for} \quad \partial_i = \frac{\partial}{\partial x^i} \ , \tag{19}$$

where $g = g(\phi^\alpha(x^a), x^i)$ is a smooth map from H^2 into the gauge group G for any fixed $x^a \in \Sigma_2$.

Let us introduce on H^2 spherical coordinates: $x^3 = \rho \cos\varphi$ and $x^4 = \rho \sin\varphi$. Using these coordinates, we impose on g the condition $g(\varphi = 0, \rho^2 \to 1) = \mathrm{Id}$ (framing) and denote by $C_0^\infty(H^2, G)$ the space of framed flat connections on H^2 given by (19). On H^2, as on a manifold with a boundary, the group of gauge transformations for any fixed $x^a \in \Sigma_2$ is defined as (see e.g. [9, 10, 11])

$$\mathcal{G}_{H^2} = \left\{ g : H^2 \to G \mid g \to \mathrm{Id} \quad \text{for} \quad \rho^2 \to 1 \right\} \ . \tag{20}$$

Hence the solution space of the equation (12) is the infinite-dimensional group $\mathcal{N} = C_0^\infty(H^2, G)$, and the moduli space of solutions is the based loop group [9, 10, 12]

$$\mathcal{M} = C_0^\infty(H^2, G)/\mathcal{G}_{H^2} = \Omega G \ . \tag{21}$$

This space can also be represented as $\Omega G = LG/G$, where $LG = C^\infty(S^1, G)$ is the loop group with the circle $S^1 = \partial H^2$ parameterized by $e^{\mathrm{i}\varphi}$.

5. Moduli space. On the group manifold (21) we introduce local coordinates ϕ^α with $\alpha = 1, 2, ...$ and recall that $\mathcal{A}_\mu$'s depend on $x^a \in \Sigma_2$ only via moduli parameters $\phi^\alpha = \phi^\alpha(x^a)$. Then moduli of gauge fields define a map

$$\phi : \Sigma_2 \to \mathcal{M} \quad \text{with} \quad \phi(x^a) = \{\phi^\alpha(x^a)\} \ . \tag{22}$$

These maps are constrained by the equations (16), (17) and (18). Since $\mathcal{A}_{H^2}$ is a flat connection for any $x^a \in \Sigma_2$, the derivatives $\partial_a \mathcal{A}_i$ have to satisfy the linearized (around $\mathcal{A}_{H^2}$) flatness condition, i.e. $\partial_a \mathcal{A}_i$ belong to the tangent space $T_\mathcal{A}\mathcal{N}$ of the space $\mathcal{N} = C_0^\infty(H^2, G)$ of framed flat connections on H^2. Using the projection $\pi : \mathcal{N} \to \mathcal{M}$ from $\mathcal{N}$ to the moduli space $\mathcal{M}$, one can decompose $\partial_a \mathcal{A}_i$ into the two parts

$$T_\mathcal{A}\mathcal{N} = \pi^* T_\mathcal{A}\mathcal{M} \oplus T_\mathcal{A}\mathcal{G} \quad \Leftrightarrow \quad \partial_a \mathcal{A}_i = (\partial_a \phi^\beta)\xi_{\beta i} + D_i \epsilon_a \ , \tag{23}$$

where $\mathcal{G}$ is the gauge group $\mathcal{G}_{H^2}$ for any fixed $x^a \in \Sigma_2$, $\{\xi_\alpha = \xi_{\alpha i}\mathrm{d}x^i\}$ is a local basis of tangent vectors at $T_\mathcal{A}\mathcal{M}$ (they form the loop Lie algebra $\Omega\mathfrak{g}$) and ϵ_a are $\mathfrak{g}$-valued gauge parameters ($D_i \epsilon_a \in T_\mathcal{A}\mathcal{G}$) which are determined by the gauge-fixing conditions

$$g^{ij} D_i \xi_{\alpha j} = 0 \quad \Leftrightarrow \quad g^{ij} D_i D_j \epsilon_a = g^{ij} D_i \partial_a \mathcal{A}_j \ . \tag{24}$$

Note also that since $\mathcal{A}_i(\phi^\alpha, x^j)$ depends on x^a only via ϕ^α, we have

$$\partial_a \mathcal{A}_i = \frac{\partial \mathcal{A}_i}{\partial \phi^\beta} \partial_a \phi^\beta \quad \stackrel{(24)}{\Longrightarrow} \quad \epsilon_a = (\partial_a \phi^\beta)\, \epsilon_\beta \ , \tag{25}$$

where the gauge parameters ϵ_β are found by solving the equations

$$g^{ij} D_i D_j \epsilon_\beta = g^{ij} D_i \frac{\partial \mathcal{A}_j}{\partial \phi^\beta} \ . \tag{26}$$

Recall that $\mathcal{A}_i$ are given explicitly by (19) and $\mathcal{A}_a$ are yet free. It is natural to choose $\mathcal{A}_a = \epsilon_a$ [5, 6] and obtain

$$\mathcal{F}_{ai} = \partial_a \mathcal{A}_i - D_i \mathcal{A}_a = (\partial_a \phi^\beta) \xi_{\beta i} = \pi_* \partial_a \mathcal{A}_i \in T_{\mathcal{A}} \mathcal{M} \ . \tag{27}$$

Thus if we know the dependence of ϕ^α on x^a then we can construct

$$(\mathcal{A}_\mu) = (\mathcal{A}_a, \mathcal{A}_i) = \Big((\partial_a \phi^\beta)\, \epsilon_\beta \ , \ g^{-1}(\phi^\alpha, x^j) \partial_i g(\phi^\beta, x^k) \Big) \ , \tag{28}$$

which are in fact the components $\mathcal{A}_\mu^{(0)} {=} \mathcal{A}_\mu(\varepsilon{=}0)$.

6. Effective action. For finding equations for $\phi^\alpha(x^a)$ we substitute (27) into (16) and see that (16) are resolved due to (24). Substituting (27) into (17), we obtain the equations

$$\frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|}\ g^{ab} \partial_b \phi^\beta \right) g^{ij} \xi_{\beta j} + g^{ab} g^{ij} (D_a \xi_{\beta j}) \partial_b \phi^\beta = 0 \ . \tag{29}$$

We should project (29) on the moduli space $\mathcal{M} = \Omega G$, metric $\mathbb{G} = (G_{\alpha\beta})$ on which is defined as

$$G_{\alpha\beta} = \langle \xi_\alpha, \xi_\beta \rangle = \int_{H^2} d\,\mathrm{vol}\ g^{ij} (\xi_{\alpha i}, \xi_{\beta j}) \ . \tag{30}$$

The projection is provided by multiplying (29) by $\langle \xi_\alpha, \cdot \rangle$ (cf. e.g. [13, 14]). We obtain

$$\begin{aligned} &\frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|}\ g^{ab} \partial_b \phi^\beta \right) \langle \xi_\alpha, \xi_\beta \rangle + g^{ab} \langle \xi_\alpha, D_a \xi_\beta \rangle \partial_b \phi^\beta \\ &= \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|}\ g^{ab} \partial_b \phi^\beta \right) G_{\alpha\beta} + \langle \xi_\alpha, \nabla_\gamma \xi_\beta \rangle g^{ab} \partial_a \phi^\gamma \partial_b \phi^\beta \\ &= G_{\alpha\sigma} \left\{ \frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|}\ g^{ab} \partial_b \phi^\sigma \right) + \Gamma^\sigma_{\beta\gamma} g^{ab} \partial_a \phi^\beta \partial_b \phi^\gamma \right\} = 0 \ , \end{aligned} \tag{31}$$

where

$$\Gamma^\sigma_{\beta\gamma} = \tfrac{1}{2}\, G^{\sigma\lambda} \left(\partial_\gamma\, G_{\beta\lambda} + \partial_\beta G_{\gamma\lambda} - \partial_\lambda G_{\beta\gamma} \right) \quad \text{with} \quad \partial_\gamma := \frac{\partial}{\partial \phi^\gamma} \ , \tag{32}$$

are the Christoffel symbols and ∇_γ are the corresponding covariant derivatives on the moduli space $\mathcal{M}$ of flat connections on H^2.

The equations

$$\frac{1}{\sqrt{|\det g_{\Sigma_2}|}} \partial_a \left(\sqrt{|\det g_{\Sigma_2}|}\ g^{ab} \partial_b \phi^\alpha \right) + \Gamma^\alpha_{\beta\gamma} g^{ab} \partial_a \phi^\beta \partial_b \phi^\gamma = 0 \tag{33}$$

are the Euler-Lagrange equations for the effective action

$$S_{\rm eff} = \int_{\Sigma_2} \mathrm{d}x^1\mathrm{d}x^2 \sqrt{-\det(g_{ab})}\, g^{cd}\, G_{\alpha\beta}\, \partial_c\phi^\alpha\partial_d\phi^\beta \tag{34}$$

obtained from the action functional (11) in the adiabatic limit $\varepsilon \to 0$; it appears from the term $(\mathcal{F}_{ai}, \mathcal{F}^{ai})$ in (11) (other terms vanish). The equations (33) are the standard sigma-model equations defining maps from Σ_2 into the based loop group ΩG.

7. Virasoro constraints. The last undiscussed equations are the constraints (18). Substituting (27) into (18), we obtain

$$g^{ij}(\xi_{\alpha i}, \xi_{\beta j})\partial_a\phi^\alpha\partial_b\phi^\beta - \tfrac{1}{2}\, g_{ab}g^{cd}g^{ij}(\xi_{\alpha i}, \xi_{\beta j})\partial_c\phi^\alpha\partial_d\phi^\beta = 0\ . \tag{35}$$

Integrating (35) over H^2 (projection on $\mathcal{M}$), we get

$$G_{\alpha\beta}\partial_a\phi^\alpha\partial_b\phi^\beta - \tfrac{1}{2}\, g_{ab}g^{cd}G_{\alpha\beta}\partial_c\phi^\alpha\partial_d\phi^\beta = 0\ . \tag{36}$$

These are equations which one will obtain from (34) by varying with respect to g_{ab}. Thus

$$T^V_{ab} = G_{\alpha\beta}\partial_a\phi^\alpha\partial_b\phi^\beta - \tfrac{1}{2}\, g_{ab}g^{cd}G_{\alpha\beta}\partial_c\phi^\alpha\partial_d\phi^\beta \tag{37}$$

is the traceless stress-energy tensor and equations (36) are the Virasoro constraints accompanying the Polyakov string action (34).

8. *B*-field. In string theory the action (34) is often extended by adding the B-field term. This term can be obtained from the topological Yang-Mills term

$$\tfrac{1}{2}\int_M \mathrm{d}^4x\, \sqrt{\det g^\varepsilon_{H_2}}\, \varepsilon_{\mu\nu\lambda\sigma}(\mathcal{F}^{\mu\nu}_\varepsilon, \mathcal{F}^{\lambda\sigma}_\varepsilon) \tag{38}$$

which in the adiabatic limit $\varepsilon \to 0$ becomes

$$\int_M \mathrm{d}^4x\, \sqrt{\det g_{H_2}}\, \varepsilon^{ab}\varepsilon^{ij}(\mathcal{F}_{ai}, \mathcal{F}_{bj}) = \int_{\Sigma_2} \mathrm{d}x^1\mathrm{d}x^2\, \varepsilon^{cd}\, B_{\alpha\beta}\, \partial_c\phi^\alpha\partial_d\phi^\beta\ , \tag{39}$$

where

$$B_{\alpha\beta} = \int_{H^2} d\,\mathrm{vol}\ \varepsilon^{ij}(\xi_{\alpha i}, \xi_{\beta j})\ . \tag{40}$$

are components of the two-form $\mathbb{B} = (B_{\alpha\beta})$ on the moduli space $\mathcal{M} = \Omega G$.

9. Remarks on superstrings. The adiabatic limit of supersymmetric Yang-Mills theories with a (partial) topological twisting on Euclidean manifold $\Sigma \times \widetilde{\Sigma}$, where Σ and $\widetilde{\Sigma}$ are Riemann surfaces, was considered in [15]. Several sigma-models with fermions on Σ (including supersymmetric ones) were obtained. Switching to Lorentzian signature and adding constraints of type (18), which were not considered in [15], one can get stringy sigma-model resembling NSR strings. However, analysis of these sigma-models demands more efforts and goes beyond the scope of our paper.

Another possibility is to consider ordinary Yang-Mills theory (11) but with Lie supergroup G as the structure group. We restrict ourselves to the N=2 super translation group with ten-dimensional

Minkowski space $\mathbb{R}^{9,1}$ as bosonic part. This super translation group can be represented as the coset [16, 17]

$$G = \mathrm{SUSY}(N{=}2)/\mathrm{SO}(9,1)\ , \tag{41}$$

with coordinates (X^α, θ^{Ap}), where $\theta^p = (\theta^{Ap})$ are two Majorana-Weyl spinors in $d = 10, \alpha = 0, ..., 9, A = 1, ..., 32$ and $p = 1, 2$. The generators of G obey the Lie superalgebra $\mathfrak{g} = \mathrm{Lie}\, G$,

$$\{\xi_{Ap}, \xi_{Bq}\} = (\gamma^\alpha C)_{AB}\delta_{pq}\xi_\alpha\ , \quad [\xi_\alpha, \xi_{Ap}] = 0\ , \quad [\xi_\alpha, \xi_\beta] = 0\ , \tag{42}$$

where γ^α are the γ-matrices in $\mathbb{R}^{9,1}$ and C is the charge conjugation matrix. On the superalgebra $\mathfrak{g}$ we introduce the standard metric

$$\langle \xi_\alpha\, \xi_\beta \rangle = \eta_{\alpha\beta}\ , \quad \langle \xi_\alpha\, \xi_{Ap} \rangle = 0 \quad \text{and} \quad \langle \xi_{Ap}\, \xi_{Bq} \rangle = 0\ , \tag{43}$$

where $(\eta_{\alpha\beta}) = \mathrm{diag}(-1, 1, ..., 1)$ is the Lorentzian metric on $\mathbb{R}^{9,1}$.

It was shown in [18] that the action functional for Yang-Mills theory on $\Sigma_2 \times H^2$ with the gauge group G, defined by (42),

$$S_\varepsilon = \frac{1}{2\pi}\int_{\Sigma_2\times H^2} \mathrm{d}^4x\, \sqrt{|\det g_{\Sigma_2}|}\,\sqrt{\det g_{H_2}}\, \Big\{\varepsilon^2\langle \mathcal{F}_{ab}\,\mathcal{F}^{ab}\rangle + 2\langle \mathcal{F}_{ai}\,\mathcal{F}^{ai}\rangle + \varepsilon^{-2}\langle \mathcal{F}_{ij}\,\mathcal{F}^{ij}\rangle\Big\} \tag{44}$$

plus the Wess-Zumino-type term

$$S_{WZ} = \frac{1}{\pi}\int_{\Sigma_3\times H^2} \mathrm{d}x^{\hat a}\wedge \mathrm{d}x^{\hat b}\wedge \mathrm{d}x^{\hat c}\wedge \mathrm{d}x^3\wedge \mathrm{d}x^4\, f_{\Gamma\Delta\Lambda}\,\mathcal{F}^\Gamma_{\hat a i}\xi^i \mathcal{F}^\Delta_{\hat b j}\xi^j \mathcal{F}^\Lambda_{\hat c k}\xi^k \tag{45}$$

yield the Green-Schwarz superstring action [17] in the adiabatic limit $\varepsilon \to 0$. Here Σ_3 is a Lorentzian manifold with the boundary $\Sigma_2 = \partial\Sigma_3$ and local coordinates $x^{\hat a}$, $\hat a = 0, 1, 2$; the structure constants $f_{\Gamma\Delta\Lambda}$ are given in [16] and $(\xi_i) = (\sin\varphi, -\cos\varphi)$ is the unit vector on H^2 running the boundary $S^1 = \partial H^2$.

10. Superstrings from $d = 3$ Yang-Mills. Here we will show that the Green-Schwarz superstrings with a worldsheet Σ_2 can also be associated with a Yang-Mills model on $\Sigma_2 \times S^1$. When the radius of S^1 tends to zero, the action of this Yang-Mills model becomes the Green-Schwarz superstring action. So, we consider Yang-Mills theory on a direct product manifold $M^3 = \Sigma_2 \times S^1$, where Σ_2 is a two-dimensional Lorentzian manifold discussed before and S^1 is the unit circle parameterized by $x^3 \in [0, 2\pi]$ with the metric tensor $g_{S^1} = (g_{33})$ and $g_{33} = 1$. As the structure group G of Yang-Mills theory we consider the super translation group in $d = 10$ auxiliary dimensions (41) with the generators (42) and the metric (43) on the Lie superalgebra $\mathfrak{g} = \mathrm{Lie}\, G$. As in (20), we impose framing over S^1, i.e. consider the group of gauge transformations equal to the identity over S^1. Coordinates on G are X^α and θ^{Ap} introduced in the previous section. The one-forms

$$\Pi^\Delta = \{\Pi^\alpha, \Pi^{Ap}\} = \{\mathrm{d}X^\alpha - \mathrm{i}\,\delta_{pq}\,\bar\theta^p\gamma^\alpha \mathrm{d}\theta^q,\ \mathrm{d}\theta^{Ap}\} \tag{46}$$

form a basis of one-forms on G [16].

By using the adiabatic approach, we deform the metric on $\Sigma_2 \times S^1$ and introduce

$$\mathrm{d}s^2_\varepsilon = g^\varepsilon_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{ab}\mathrm{d}x^a\mathrm{d}x^b + \varepsilon^2(\mathrm{d}x^3)^2\ , \tag{47}$$

where $\varepsilon \in [0,1]$ is a real parameter, $a,b = 1,2$, $\mu,\nu = 1,2,3$. This is equivalent to the consideration of the circle S^1_ε of radius ε. It is assumed that for the fields $\mathcal{A}_\mu$ and $\mathcal{F}_{\mu\nu}$ there exist limits $\lim_{\varepsilon\to 0}\mathcal{A}_\mu$ and $\lim_{\varepsilon\to 0}\mathcal{F}_{\mu\nu}$. Indices are raised by $g^{\mu\nu}_\varepsilon$ and we have

$$\mathcal{F}^{ab}_\varepsilon = g^{ac}_\varepsilon g^{bd}_\varepsilon \mathcal{F}_{cd} = \mathcal{F}^{ab} \ , \quad \mathcal{F}^{a3}_\varepsilon = g^{ac}_\varepsilon g^{33}_\varepsilon \mathcal{F}_{c3} = \varepsilon^{-2}\mathcal{F}^{a3} \ , \tag{48}$$

where indices in $\mathcal{F}^{\mu\nu}$ are raised by the non-deformed metric tensor.

We consider the Yang-Mills action of the form

$$S_\varepsilon = \int_{M^3} \mathrm{d}^3x \sqrt{|\det g_{\Sigma_2}|}\ \left\{\frac{\varepsilon^2}{2}\langle \mathcal{F}_{ab}\,\mathcal{F}^{ab}\rangle + \langle \mathcal{F}_{a3}\,\mathcal{F}^{a3}\rangle\right\} \ , \tag{49}$$

which for $\varepsilon = 1$ coincides with the standard Yang-Mills action. Variations with respect to $\mathcal{A}_\mu$ and g_{ab} yield the equations

$$\varepsilon^2 D_a\mathcal{F}^{ab} + D_3\mathcal{F}^{3b} = 0 \ , \quad D_a\mathcal{F}^{a3} = 0 \ , \tag{50}$$

$$T^\varepsilon_{ab} = \varepsilon^2\left(g^{cd}\langle \mathcal{F}_{ac}\,\mathcal{F}^{bd}\rangle - \tfrac{1}{4}\,g_{ab}\langle \mathcal{F}_{cd}\,\mathcal{F}^{cd}\rangle\right) + \langle \mathcal{F}_{a3}\,\mathcal{F}_{b3}\rangle - \tfrac{1}{2}\,g_{ab}\langle \mathcal{F}_{c3}\,\mathcal{F}^{c3}\rangle \ . \tag{51}$$

In the adiabatic limit $\varepsilon \to 0$ equations (50), (51) become

$$D_3\mathcal{F}^{3b} = 0 \ , \quad D_a\mathcal{F}^{a3} = 0 \ , \tag{52}$$

$$T^0_{ab} = \langle \mathcal{F}_{a3}\,\mathcal{F}_{b3}\rangle - \tfrac{1}{2}\,g_{ab}\langle \mathcal{F}_{c3}\,\mathcal{F}^{c3}\rangle \ . \tag{53}$$

Notice that as a function of $x^3 \in S^1$, the field $\mathcal{A}_3$ belongs to the loop algebra $L\mathfrak{g} = \mathfrak{g} \oplus \Omega\mathfrak{g}$, where $\Omega\mathfrak{g}$ is the Lie superalgebra of the based loop group ΩG. Let us denote by $\mathcal{A}^0_3$ the zero-mode in the expansion of $\mathcal{A}_3$ in $\exp(\mathrm{i}\,x^3) \in S^1$ (Wilson line). The generic $\mathcal{A}_3$ can be represented in the form

$$\mathcal{A}_3 = h^{-1}\mathcal{A}^0_3 h + h^{-1}\partial_3 h \ , \tag{54}$$

where G-valued function h depends on x^a and x^3. For fixed $x^a \in \Sigma_2$ one can choose $h \in \Omega G$=Map$(S^1, G)/G$. We denote by $\mathcal{N}$ the space of all $\mathcal{A}_3$ given by (54) and define the projection $\pi : \mathcal{N} \to G$ on the space G parametrizing $\mathcal{A}^0_3$ since we want to keep only $\mathcal{A}^0_3$ in the limit $\varepsilon \to 0$. We denote by Q the fibres of the projection π.

In the adiabatic approach it is assumed that $\mathcal{A}^0_3$ depends on $x^a \in \Sigma_2$ only via the moduli parameters $(X^\alpha, \theta^{Ap}) \in G$. Therefore, the moduli define the maps

$$(X, \theta^p) : \quad \Sigma_3 \to G \tag{55}$$

which are not arbitrary, they are constrained by the equations (52), (53). The derivatives $\partial_a\mathcal{A}_3$ of $\mathcal{A}_3 \in \mathcal{N}$ belong to the tangent space $T_{\mathcal{A}_3}\mathcal{N}$ of the space $\mathcal{N}$. Using the projection $\pi : \mathcal{N} \to G$, one can decompose $\partial_a\mathcal{A}_3$ into two parts

$$T_{\mathcal{A}_3}\mathcal{N} = \pi^* T_{\mathcal{A}^0_3} G \oplus T_{\mathcal{A}_3} Q \quad \Leftrightarrow \quad \partial_a\mathcal{A}_3 = \Pi^\Delta_a \xi_{\Delta 3} + D_3\epsilon_a \ , \tag{56}$$

where $\Delta = (\alpha, Ap)$ and

$$\Pi^\alpha_a := \partial_a X^\alpha - \mathrm{i}\,\delta_{pq}\,\bar{\theta}^p\gamma^\alpha\partial_a\theta^q \ , \quad \Pi^{Ap}_a := \partial_a\theta^{Ap} \ . \tag{57}$$

In (56), ϵ_a are $\mathfrak{g}$-valued parameters ($D_3\epsilon_a \in T_{\mathcal{A}_3}Q$) and the vector fields $\xi_{\Delta 3}$ on G can be identified with the generators $\xi_\Delta = (\xi_\alpha, \xi_{Ap})$ of G.

On $\xi_{\Delta 3}$ we impose the gauge fixing condition

$$D_3\xi_{\Delta 3} = 0 \quad \overset{(56)}{\Longrightarrow} \quad D_3D_3\epsilon_a = D_3\partial_a\mathcal{A}_3 \ . \tag{58}$$

Recall that $\mathcal{A}_3$ is fixed by (54) and $\mathcal{A}_a$ are yet free. In the adiabatic approach one chooses $\mathcal{A}_a = \epsilon_a$ (cf. [5, 6]) and obtains

$$\mathcal{F}_{a3} = \partial_a\mathcal{A}_3 - D_3\mathcal{A}_a = \Pi_a^\Delta\xi_{\Delta 3} \in T_{\mathcal{A}_3^0}G \ . \tag{59}$$

Substituting (59) into the first equation in (52), we see that they are resolved due to (58). Substituting (59) into the action $S_0 = \lim_{\varepsilon\to 0} S_\varepsilon$ given by (49) and integrating over x^3, we obtain the effective action

$$S_0 = 2\pi \int_{\Sigma_2} \mathrm{d}^2x \, \sqrt{|\det g_{\Sigma_2}|} \, g^{ab} \, \Pi_a^\alpha \, \Pi_b^\beta \, \eta_{\alpha\beta} \ , \tag{60}$$

which coincides with the kinetic part of the Green-Schwarz superstring action [17]. One can show (cf. [14]) that the second equations in (52) are equivalent to the Euler-Lagrange equations for (X^α, θ^{Ap}) following from (60). Finally, substituting (59) into (53), we obtain the equations

$$\Pi_a^\alpha \, \Pi_b^\beta \, \eta_{\alpha\beta} - \tfrac{1}{2} \, g_{ab} \, g^{cd} \, \Pi_c^\alpha \, \Pi_d^\beta \, \eta_{\alpha\beta} = 0 \tag{61}$$

which can also be obtained from (60) by variation of g^{ab}.

For getting the full Green-Schwaz superstring action one should add to (60) a Wess-Zumino-type term which is described as follows [16, 17]. One should consider a Lorentzian 3-manifold Σ_3 with the boundary $\Sigma_2 = \partial\Sigma_3$ and coordinates $x^{\hat a}$, $\hat a = 0, 1, 2$. On Σ_3 one introduces the 3-form [16]

$$\Omega_3 = \mathrm{i}\,\mathrm{d}x^{\hat a}\Pi_{\hat a}^\alpha \wedge (\check{\mathrm{d}}\bar\theta^1\gamma^\beta \wedge \check{\mathrm{d}}\theta^1 - \check{\mathrm{d}}\bar\theta^2\gamma^\beta \wedge \check{\mathrm{d}}\theta^2)\,\eta_{\alpha\beta} = \check{\mathrm{d}}\Omega_2 \ , \tag{62}$$

where

$$\Omega_2 = -\mathrm{i}\check{\mathrm{d}}X^\alpha \wedge (\bar\theta^1\gamma^\beta\check{\mathrm{d}}\theta^1 - \bar\theta^2\gamma^\beta\check{\mathrm{d}}\theta^2) \quad \text{with} \quad \check{\mathrm{d}} = \mathrm{d}x^{\hat a}\frac{\partial}{\partial x^{\hat a}} \ .$$

Then the term

$$S_{WZ} = \int_{\Sigma_3} \Omega_3 = \int_{\Sigma_2} \Omega_2 \tag{63}$$

is added to (60) with a proper coefficient $\varkappa$ and $S_{GS} = S_0 + \varkappa S_{WZ}$ is the Green-Schwarz action for the superstrings of type I, IIA and IIB.

To get (63) from Yang-Mills theory we consider the manifold $\Sigma_3 \times S^1$ and notice that in addition to (59) we now have the components

$$\mathcal{F}_{03} = \Pi_0^\Delta\xi_{\Delta 3} = (\partial_0 X^\alpha - \mathrm{i}\,\delta_{pq}\bar\theta^p\gamma^\alpha\partial_0\theta^q)\xi_{\alpha 3} + (\partial_0\theta^{Ap})\xi_{Ap3}. \tag{64}$$

Introduce one-forms $F_3 := \mathcal{F}_{\hat a 3}\mathrm{d}x^{\hat a}$ on Σ_3, where $\mathcal{F}_{\hat a 3}(\varepsilon)$ are general Yang-Mills fields on $\Sigma_3 \times S^1$ which take the form (59),(64) only in the limit $\varepsilon \to 0$, and consider the functional

$$S_{WZ}^{YM} = \int_{\Sigma_3\times S^1} f_{\Delta\Lambda\Gamma} \, F_3^\Delta \wedge F_3^\Lambda \wedge F_3^\Gamma \wedge \mathrm{d}x^3 \ , \tag{65}$$

where the explicit form of the constant $f_{\Delta\Lambda\Gamma}$ can be found in [16]. Therefore, the Yang-Mills action (49) plus (65) in the adiabatic limit $\varepsilon \to 0$ becomes the Green-Schwarz action. This result can be considered as a generalization of the Green result [19] who derived the superstring theory in a fixed gauge from Chern-Simons theory on $\Sigma_2 \times \mathbb{R}$.

11. Concluding remarks. We have shown that bosonic strings and Green-Schwarz superstrings can be obtained via the adiabatic limit of Yang-Mills theory on manifolds $\Sigma_2 \times H^2$ with a Wess-Zumino-type term. Notice that the constraint equations (15) on the Yang-Mills energy momentum tensor with $\varepsilon > 0$ are important for restoring the unitarity of Yang-Mills theory on $\Sigma_2 \times H^2$. More interestingly, the same result is also obtained by considering Yang-Mills theory on three-dimensional manifolds $\Sigma_2 \times S^1_\varepsilon$ with the radius of the circle S^1_ε given by $\varepsilon \in [0, 1]$. For $\varepsilon \neq 0$ we have well-defined quantum Yang-Mills theory on $\Sigma_2 \times S^1_\varepsilon$. For $\varepsilon \to 0$ we get superstring theories. This raises hopes that various results for superstring theories can be obtained from results of the associated Yang-Mills theory on $\Sigma_2 \times S^1_\varepsilon$.

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